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Renormalization group in the infinite-dimensional turbulence: third-order results

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Abstract

The field theoretic renormalization group is applied to the stochastic Navier–Stokes equation with the stirring force correlator of the form $k^{4-d-2\varepsilon}$ in the d-dimensional space, in connection with the problem of construction of the $1/d$ expansion for the fully developed fluid turbulence beyond the scope of the standard ε expansion. It is shown that in the large-d limit the number of the Feynman diagrams for the Green function (linear response function) decreases drastically, and the technique of their analytical calculation is developed. The main ingredients of the renormalization group approach—the renormalization constant, the β function and the ultraviolet correction exponent ω, are calculated to order ε^3 (three-loop approximation). The two-point velocity–velocity correlation function, the Kolmogorov constant C_K in the spectrum of turbulent energy and the inertial-range skewness factor $\mathcal{S}$ are calculated in the large-d limit to the third order of the ε expansion. Surprisingly enough, our results for C_K are in reasonable agreement with the existing experimental estimates.

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1. Introduction

One of the most interesting open problems inherited by modern theoretical physics from the 20th century is that of the description of fully developed hydrodynamic turbulence on the basis of a microscopic model and within a consistent perturbation scheme [1]. A challenge which is still waiting to be answered is the derivation of anomalous scaling behaviour of the velocity correlation functions from first principles and calculation of the corresponding anomalous exponents within a regular perturbation theory, analogous to the ε or $1/N$ expansions of the critical exponents in the theory of second-order phase transitions.

The first difficulty is that the ordinary perturbation theory for the stirred (stochastic) Navier–Stokes (NS) equation (that is, the expansion in the nonlinearity) is in fact the expansion in the Reynolds number, a parameter which tends to infinity for the fully developed turbulence. Hence the necessity to rearrange (to sum up) the naive perturbation series. A similar problem is well known in the theory of critical behaviour, where it had been solved a long ago by means of the renormalization group (RG) and the self-consistency ('bootstrap') diagrammatic equations [2, 3]. The first way naturally leads to the famous ε expansion (where $\varepsilon = 4 - d$ is the deviation of the spatial dimension d from its upper critical value $d = 4$), the second leads to the alternative $1/N$ expansion (where N is the number of components of the corresponding order parameter). So far, however, those methods have had relatively limited success when applied to the problem of anomalous scaling in fluid turbulence.

An important difference is that the turbulence or, better to say, the stochastic NS equation, has no upper critical dimension, and the parameter ε in the standard RG approach has completely different meaning. Namely, the correlation function of the random stirring force that provides the energy supply to the system is taken in the power-law form [4]

$$\langle ff \rangle \propto k^{4-d-2\varepsilon}, \tag{1.1}$$

where k is the wave number; more precisely, see [3, 5, 6] and section 2.

The physical value $\varepsilon = 2$ corresponds to the energy pumping by the largest scales, $\langle ff \rangle \propto \delta(\mathbf{k})$, while d remains a free parameter and can be varied independently of ε. The common point with the models of critical behaviour is that, in both cases, the limit $\varepsilon \to 0$ corresponds to a logarithmic (exactly renormalizable) theory and ε serves as the formal small expansion parameter in the RG approach. For this reason, we use the same symbol ε for the stochastic NS problem (where it is also sometimes denoted as $\varepsilon = y/2$).

The results of the RG analysis of this model are reliable and internally consistent for asymptotically small ε, while the possibility of their extrapolation to the physical value $\varepsilon = 2$ and thus their relevance for the real fluid turbulence is far from obvious. Of course, the physical value of $\varepsilon = 4 - d$ in the RG theory of critical phenomena is not small, either. But, no qualitative changeover in the behaviour of the system is expected when ε increases from the region $\varepsilon \ll 1$ to real values $\varepsilon \sim 1$, and the possibility of this extrapolation is usually not disputed. The situation in the RG approach to the NS model (1.1) appears different. New physical effects are encountered as ε grows, and they can be easily lost or misrepresented if the ε expansion is too naively applied. One of them is related to the so-called sweeping effects, the transport of small turbulent eddies as a whole by large-scale ones. The sweeping leads to strong dependence of the velocity correlation functions of the integral (external) turbulence scale L. The other effect is the crossover from the Kolmogorov 'K41'scaling to the anomalous (multi-)scaling—singular dependence of the Galilean invariant quantities (like e.g. equal-time structure functions) on L, characterized by an infinite set of independent anomalous exponents. These effects lead to infrared (IR) divergences in the diagrams of perturbation theory which, formally speaking, manifest themselves as poles at some finite values of ε. Such poles and their physical interpretation were discussed in a number of papers; see [7–10].

The aforementioned crossovers have no analogue in the models of critical behaviour, but such IR poles are present in their diagrams, too. Moreover, they approach closer and closer to the origin $\varepsilon = 0$ when the order of the perturbation theory (complexity of the corresponding diagrams) increases. The correct resummation of these IR singularities is accomplished by the short-distance operator-product expansion (SDE); it shows that the finite limit $L \to \infty$ (where L is some IR scale) in the correlation functions exists and the singular-in-L terms give only subleading corrections to the scaling behaviour [2, 3]. Thus, the existence of IR singularities at finite values of ε does not hinder the use of the RG technique and the ε expansion for the description of critical behaviour.

The SDE technique is equally applied to the stochastic NS problem. The distinguishing feature, specific to models of turbulence, is the existence in the corresponding SDE of composite fields ('operators') with *negative* scaling dimensions. Such operators were named 'dangerous' in [11] because their contributions to the OPE diverge for $L \to \infty$. The summation of the most singular contributions coming from the operators v^n, powers of the velocity field, was performed in [11], the generalization to the case of a time-dependent large-scale field is given in [12]. This gives the adequate description of the sweeping effects within the RG formalism; see also the general discussion in [5, 6].

According to the SDE scenario, anomalous multiscaling in the structure functions can be related to the existence of Galilean invariant dangerous operators. This idea was successfully realized for the model of a scalar impurity field passively advected by the Gaussian velocity field with the given correlation function $\propto \delta(t-t')/k^{d+\varepsilon}$, known as the Obukhov–Kraichnan rapid-change model. In the zero-mode approach, which can be viewed as a variant of the self-consistency equations, the anomalous exponents were derived analytically to order $O(\varepsilon)$ [13] and $O(1/d)$ [14]; see also [15] for a detailed discussion. In the RG and SDE approach, the exponents are identified with the scaling dimensions of composite operators built of the scalar gradients (namely, powers of the dissipation rate of the scalar field fluctuations, and their tensor analogues) and calculated within the expansion in the exponent ε (also denoted by ξ) to order ε^3 (three-loop approximation of the RG); see the original works [16, 17] and the review paper [18] for generalizations and more references.

In the stochastic NS problem, however, all the critical dimensions are strictly positive for small ε and can become negative only for some finite values of ε. Therefore, dangerous invariant operators cannot be identified within the ε expansions and it is desirable to construct an alternative perturbation scheme valid for finite ε. Attempts were made to modify the model by introducing N replicas of the velocity field and to construct an expansion in $1/N$ [19], but such modifications were inconsistent with the Galilean symmetry [20]; see also discussion [21, 22] for the case of discrete shell models.

Thus we naturally return to an old idea of the expansion in $1/d$, which has repeatedly been introduced in various contexts of turbulence [23–28] and looks very attractive for a few reasons.

One can hope that in the limit $d \to \infty$ intermittency and anomalous scaling disappear or acquire a simple 'calculable' form and the finite-dimensional turbulence can be studied within the expansion around this 'solvable' limit [24]. However, in contrast to the $O(N)$-symmetric model of the critical behaviour, where the limit $N \to \infty$ is described in a closed form by the exactly solvable spherical model [29], no drastic simplifications were found in [24] for the infinite-dimensional turbulence: all classes of diagrams for the velocity correlation function (for nonstationary perturbation theory with the Taylor expansion in time and for renormalized perturbation theory with skeleton diagrams with dressed lines) survive in the large-d limit, and the incompressibility condition (and thus the nonlocal pressure effects) remain important. Much later it was argued (on the basis of a certain SDE-motivated ansatz for dissipative terms) that the K41 theory becomes exact and the multiscaling indeed disappears for $d = \infty$ [25], as it also happens for the Obukhov–Kraichnan model [23]. What is more, for the latter it was possible to find the $O(1/d)$ contribution to the anomalous exponents [14]. Numerical simulation of the passive scalar advection in the Obukhov–Kraichnan model for large d (up to $d = 30$) was performed in [30]. However, the systematic expansion in $1/d$ has not been yet constructed for that model, let alone the stochastic NS equation.

The key idea of the present paper is to combine the large-d limit with the RG approach and the expansion in ε. It was noted earlier in [27, 28] that taking the limit $d \to \infty$ leads to serious simplifications in the RG calculations. In a very important paper [27], scaling dimensions of all the powers of the local energy dissipation rate were calculated for $d = \infty$ to first order in ε. For a finite d, such calculation becomes an extremely daunting task due to the mixing of operators in the renormalization, and it was not completely performed even for the second power of the dissipation rate; see [31]. The scaling dimensions calculated in [27] appear positive for $\varepsilon < 2$ and vanish at the physical value $\varepsilon = 2$, in agreement with the arguments of [24, 25] that the K41 scaling exactly holds for $d = \infty$. It is then possible that the $O(\varepsilon/d)$ correction to the result of [27] will make the dimensions negative and the anomalous scaling will be established as a fact reliable within the double expansion in ε and $1/d$ (we recall that the powers of the dissipation rate are the operators 'responsible' for the anomalous scaling in the Obukhov–Kraichnan model [16–18]). For the vector analogue of the rapid-change model, where the calculation of the anomalous exponents also becomes rather involved already in the $O(\varepsilon)$ order due to the mixing of composite fields, additional expansion in $1/d$ also leads to serious simplification [28].

In this paper we systematically investigate the infinite-d limit within the RG framework. Although we were not able to exactly solve the problem for $d = \infty$ nor to construct an analogue of the 'spherical model', we obtained a number of new results which we consider interesting and which, as we believe, can give some hints for the further study of the large-d behaviour.

In section 2 we briefly recall the field theoretic formulation, renormalization and the main findings of the RG approach for the stirred NS equation. A more detailed specification of the limit $d \to \infty$ is given. In section 3 we consider the linear response (Green) function and show that, for $d \to \infty$, lots of diagrams vanish and the remaining ones simplify drastically. This allows us to analytically calculate in the three-loop approximation the key ingredients of the RG approach: the renormalization constant, the β function, the coordinate of the fixed point and the ultraviolet (UV) correction exponent ω (for finite d, only two-loop results for these quantities are known, and they involve *numerical* calculations of the corresponding diagrams [10]). Some speculations are also made about the hypothetical expressions for these quantities beyond the scope of the ε expansion.

The two-point velocity–velocity correlation function is analysed in section 4. Here, no remarkable simplifications in the diagrams occur in the limit $d \to \infty$, in agreement with the observations made earlier in [24]. However, we propose some trick which allows us to perform the analytical calculation of the pair correlator in the two-loop approximation (which, in orders of the coupling constant, corresponds to the three-loop approximation for the Green function). Although the results can be interpreted *a posteriori* in terms of the steepest-descent method, the practical use of the latter beyond the one-loop approximation is hardly possible due to complexity of the corresponding integrals.

These results are exploited in section 5. There, we calculate to the third order in ε the Kolmogorov constant C_K in the spectrum of turbulent energy and the inertial-range skewness factor $\mathcal{S}$. An important point here is not only the inclusion of the third-order correction, but also the derivation of C_K through a universal (in the sense of the theory of critical behaviour) quantity. This approach was proposed earlier in [10], where C_K was calculated (for $d = 3$) to the second order in ε. It allows one to obviate the main shortcoming of earlier calculations of C_K: the intrinsic ambiguities in the corresponding ε expansions (see e.g. [32] and the discussion in [6]). The results are briefly summarized in the conclusion.

2. Stochastic Navier–Stokes equation, choice of the random force and the field theoretic formulation

As the microscopic dynamical model of the fully developed, homogeneous, isotropic turbulence of an incompressible viscous fluid one usually takes the stochastic NS equation with a random stirring force

$$\nabla_t v_i = \nu_0 \partial^2 v_i - \partial_i \mathcal{P} + f_i, \qquad \nabla_t = \partial_t + v_i \partial_i, \tag{2.1}$$

where v_i is the transverse (divergence-free, due to the incompressibility condition $\partial_i v_i = 0$) velocity field, $\mathcal{P}$ and f_i are the pressure and the transverse random force per unit mass (all these quantities depend on $x = \{t, \mathbf{x}\}$), ν_0 is the kinematic viscosity coefficient, ∂^2 is the Laplacian and ∇_t is the Lagrangian derivative. Problem (2.1) is studied on the entire t-axis and is augmented by the retardation condition and the condition that v_i vanishes for $t \to -\infty$.

We assume for f a Gaussian distribution with zero mean and correlation function

$$\langle f_i(x) f_j(x') \rangle = \frac{\delta(t-t')}{(2\pi)^d} \int \mathrm{d}\mathbf{k}\, P_{ij}(\mathbf{k}) d_f(k) \exp[\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')], \tag{2.2}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector, $d_f(k)$ is some function of the wave number $k = |\mathbf{k}|$ and model parameters and d is the dimension of the $\mathbf{x}$ space. The time decorrelation of the random force guarantees Galilean invariance of the stochastic problem (2.1) and (2.2).

Let us specify the form of the function $d_f(k)$ in the correlator (2.2) used in the RG theory of turbulence. Physically, the random force models the injection of energy into the system due to interaction with the large-scale motions. Idealized injection by infinitely large vortices corresponds to

$$d_f(k) = 2(2\pi)^d \mathcal{E} \delta(\mathbf{k})/(d-1), \tag{2.3}$$

where $\mathcal{E}$ is the average power of the injection (equal to the average dissipation rate) and the amplitude factor comes from the exact relation

$$\mathcal{E} = \frac{(d-1)}{2(2\pi)^d} \int \mathrm{d}\mathbf{k}\, d_f(k). \tag{2.4}$$

On the other hand, for the use of the standard RG technique it is important that the function $d_f(k)$ have a power-law behaviour at large k. This condition is satisfied if $d_f(k)$ is chosen in the form

$$d_f(k) = D_0 k^{4-d-2\varepsilon}, \tag{2.5}$$

where $D_0 > 0$ is the amplitude factor and $\varepsilon > 0$ is the exponent with the physical value $\varepsilon = 2$. This can be explained with the aid of the well-known power-law representation of the d-dimensional δ function

$$\begin{aligned} \delta(\mathbf{k}) &= \lim_{\varepsilon\to 2} \frac{1}{(2\pi)^d} \int \mathrm{d}\mathbf{x} (\Lambda x)^{2\varepsilon-4} \exp[\mathrm{i}(\mathbf{kx})] \\ &= S_d^{-1} k^{-d} \lim_{\varepsilon\to 2} [(4-2\varepsilon)(k/\Lambda)^{4-2\varepsilon}], \end{aligned} \tag{2.6}$$

with some UV momentum scale Λ. Here and below we denote

$$S_d = 2\pi^{d/2}/\Gamma(d/2), \qquad \bar{S}_d = S_d/(2\pi)^d, \tag{2.7}$$

where S_d is the surface area of the unit sphere in d-dimensional space and $\Gamma(\cdots)$ is the Euler gamma function. For $\varepsilon \to 2$, function (2.5) turns to the ideal injection (2.3) if the amplitude D_0 is related to $\mathcal{E}$ as

$$D_0 \to \frac{4(2-\varepsilon)\Lambda^{2\varepsilon-4}}{\bar{S}_d (d-1)} \mathcal{E} \qquad \text{for} \quad \varepsilon \to 2. \tag{2.8}$$

As already mentioned, in the RG approach to the problems (2.1), (2.2) and (2.5) the exponent ε plays the part analogous to that played by $4-d$ in Wilson's theory of critical phenomena, while d remains a free parameter. A more realistic model is

$$d_f(k) = D_0 k^{4-d-2\varepsilon} h(m/k), \qquad h(0) = 1, \tag{2.9}$$

where $m = 1/L$ is the reciprocal of the integral turbulence scale L and $h(m/k)$ is some well-behaved function that provides the IR regularization. Its specific form is unessential; we shall always use the sharp cutoff

$$h(m/k) = \Theta(k-m), \tag{2.10}$$

which is the most convenient choice from the calculational viewpoints; $\Theta(\cdots)$ being the Heaviside step function.

According to the general theorem [33], stochastic problem (2.1), (2.2) is equivalent to the field theoretic model of the doubled set of fields $\Phi = \{\mathbf{v}', \mathbf{v}\}$ with action functional

$$S(\Phi) = v' D_v v'/2 + v'[-\nabla_t + \nu_0 \partial^2]v, \tag{2.11}$$

where D_v is the correlation function (2.2) of the random force f_i with the function $d_f(k)$ from (2.9) and all the required integrations over $x = \{t, \mathbf{x}\}$ and summations over the vector indices are understood, for example,

$$v'(v\partial)v = \int \mathrm{d}t \int \mathrm{d}\mathbf{x}\, v_i'(v_j \partial_j) v_i.$$

The auxiliary vector field is also transverse, $\partial_i v_i' = 0$, which allows us to omit the pressure term on the right-hand side of equation (2.11), as becomes evident after the integration by parts:

$$\int \mathrm{d}t \int \mathrm{d}\mathbf{x}\, v_i' \partial_i \mathcal{P} = -\int \mathrm{d}t \int \mathrm{d}\mathbf{x}\, \mathcal{P}(\partial_i v_i') = 0.$$

Of course, this does not mean that the pressure contribution can simply be neglected: the field v' acts as the transverse projector and selects the transverse part of the expressions to which it is contracted in (2.11).

Formulation (2.11) means that statistical averages of random quantities in the original stochastic problem (2.1), (2.2) can be represented as functional averages with the weight $\exp S(\Phi)$, and the generating functionals of total $[G(A)]$ and connected $[W(A)]$ correlation functions of the problem are represented by the functional integral

$$G(A) = \exp W(A) = \int \mathcal{D}\Phi \exp[S(\Phi) + A\Phi] \tag{2.12}$$

with arbitrary sources $A = \{A^{v'}, A^{v}\}$ in the linear form $A\Phi = \sum_\Phi \int \mathrm{d}x\, A^\Phi(x)\Phi(x)$.

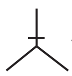
The model (2.11) corresponds to a standard Feynman diagrammatic technique; the bare propagators (lines in the diagrams) in the time–momentum (t–$\mathbf{k}$) representation have the forms

$$\langle v_i(t) v_j'(t')\rangle_0 = \Theta(t-t') \exp\{-\nu_0 k^2 (t-t')\} P_{ij}(\mathbf{k}) =$$


$$\langle v_i(t) v_j(t')\rangle_0 = \frac{d_f(k)}{2\nu_0 k^2} \exp\{-\nu_0 k^2 |t-t'|\} P_{ij}(\mathbf{k}) = \tag{2.13}$$

$$\langle v_i'(t) v_j'(t')\rangle_0 = 0,$$

with $d_f(k)$ from (2.9) and the Heaviside step function $\Theta(\cdots)$. The interaction in (2.11) corresponds to the triple vertex $-v'(v\partial)v = v_i' V_{ijs}^S v_j v_s/2$ with vertex factor

$$V_{ijs}^S = \mathrm{i}(k_j \delta_{is} + k_s \delta_{ij}) = \tag{2.14}$$


where $\mathbf{k}$ is the momentum argument of the field v'.

It is convenient to introduce the new parameter ('coupling constant') by the relation

$$g_0 = D_0 \big/ \nu_0^3, \tag{2.15}$$

so that $g_0 \propto \Lambda^{2\varepsilon}$ with the UV momentum scale Λ coming from (2.6). Thus the model (2.11) becomes logarithmic (the coupling constant becomes dimensionless) at $\varepsilon = 0$, and the UV divergences manifest themselves as the poles in ε in the correlation functions of the fields $\Phi = \{v, v'\}$. Dimensional analysis (power counting) shows that superficial UV divergences, whose removal requires counterterms, can be present only in the 1-irreducible correlation functions $\langle v'v \rangle_{\text{1-ir}}$ and the corresponding counterterms reduce to the forms $v'\partial^2 v$, $v'\partial_t v$ and $v'(v\partial)v$. In fact, in the model (2.11) there are fewer possible counterterms than allowed by the naive power counting. The symbol ∂ at the vertex in (2.11) can be moved onto the field v' using the integration by parts, which means that the counterterms to the 1-irreducible functions must contain at least one spatial derivative per each field v'. This excludes the structure $v'\partial_t v$. The Galilean symmetry requires that the counterterms $v'\partial_t v$ and $v'(v\partial)v$ must form the invariant combination $v'\nabla_t v$, which excludes the structure $v'(v\partial)v$. In the special case $d = 2$ a new UV divergence appears in the 1-irreducible function $\langle v'v' \rangle_{\text{1-ir}}$. Since we are interested in the large-d limit here, we can simply ignore that divergence.

Then the inclusion of the only remaining counterterm $v'\partial^2 v$ in the action functional (2.11) is reproduced by the multiplicative renormalization of the parameters ν_0 and g_0 with the only independent renormalization constant Z_ν:

$$\nu_0 = \nu Z_\nu, \qquad g_0 = g\mu^{2\varepsilon} Z_g, \qquad Z_g = Z_\nu^{-3} \qquad \big(D_0 = g_0 \nu_0^3 = g\mu^{2\varepsilon}\nu^3\big). \tag{2.16}$$

Here μ is the reference mass in the minimal subtraction (MS) scheme, which we always use in what follows, g and ν are renormalized analogues of the bare parameters g_0 and ν_0, and $Z = Z(g, \varepsilon, d)$ are the renormalization constants. No renormalization of the fields and the IR scale $m_0 = m$ is needed, i.e., $Z_\Phi = 1$ for all Φ and $Z_m = 1$. The renormalized action has the form

$$S(\Phi) = v' D_v v'/2 + v'[-\nabla_t + \nu Z_\nu \partial^2] v, \tag{2.17}$$

where the amplitude D_0 in D_v is expressed in renormalized parameters using the last relation from (2.16).

In the MS scheme the renormalization constants have the form '1 + only poles in ε', in particular,

$$Z_\nu = 1 + \sum_{k=1}^{\infty} a_k(g)\varepsilon^{-k} = 1 + \sum_{n=1}^{\infty} g^n \sum_{k=1}^{n} a_{nk}\varepsilon^{-k}, \tag{2.18}$$

where the coefficients a_{nk} depend only on d.

Since the fields are not renormalized, their renormalized correlation functions G^R coincide with their unrenormalized analogues $G = \langle \Phi \cdots \Phi \rangle$; the only difference is in the choice of variables and in the form of perturbation theory (in g instead of g_0): $G^R(g, \nu, \mu, m, \ldots) = G(g_0, \nu_0, m_0, \ldots)$. Here the dots stand for other arguments like coordinates, times, momenta and so on. We use $\widetilde{\mathcal{D}}_\mu$ to denote the differential operator $\mu\partial_\mu$ for fixed bare parameters g_0, ν_0, m_0 and operate on both sides of that relation with it. This gives the basic differential RG equation:

$$\mathcal{D}_{RG} G^R(g, \nu, \mu, m, \ldots) = 0, \qquad \mathcal{D}_{RG} = \mathcal{D}_\mu + \beta(g)\partial_g - \gamma_\nu(g)\mathcal{D}_\nu, \tag{2.19}$$

where $\mathcal{D}_{RG}$ is the operation $\widetilde{\mathcal{D}}_\mu$ expressed in renormalized variables, $\mathcal{D}_x = x\partial_x$ for any variable x, and the RG functions (the anomalous dimension γ_ν and the β function) are defined as

$$\begin{aligned} \gamma_\nu(g) &= \widetilde{\mathcal{D}}_\mu \ln Z_\nu = \beta(g, \varepsilon)\partial_g \ln Z_\nu, \\ \beta(g, \varepsilon) &= \widetilde{\mathcal{D}}_\mu g = g[-2\varepsilon + 3\gamma_\nu(g)], \end{aligned} \tag{2.20}$$

the relation between β and γ_ν results from the definitions and the last relation in (2.16). Combining the two relations in (2.20) and substituting (2.18) gives

$$\gamma_\nu(g) = -2\mathcal{D}_g a_1(g) + \text{the terms containing only poles in } \varepsilon, \tag{2.21}$$

with $a_1(g)$ from (2.18). From the UV finiteness of the renormalized functions G^R it follows that the pole terms in (2.21) cancel each other, which eventually gives

$$\gamma_\nu(g) = -2\mathcal{D}_g a_1(g) = -2\sum_{n=1}^{\infty} n a_{n1} g^n \tag{2.22}$$

with $a_1(g)$ and a_{n1} defined in (2.18). The cancellation occurs due to the fact that the coefficients a_{nk} with $k > 1$ are algebraically related to a_{n1}, the coefficients in front of the first-order poles, for example, $a_{22} = a_{21}^2$. Such relations can be used to check the results of the practical calculations.

The one-loop result

$$a_{11} = -(d-1)\bar{S}_d/8(d+2) \tag{2.23}$$

with $\bar{S}_d$ from (2.7) is well known, while the two-loop coefficient a_{21} was calculated (for a few values of d, including $d = 3$ and in the limits $d \to 2$ and $d \to \infty$) in [10]. In the latter limit, which we are interested in here, one obtains

$$a_{21} = a_{11}^2\left\{-\tfrac{1}{2} + O(1/d)\right\}. \tag{2.24}$$

From the first-order result (2.23) it follows that the β function has a nontrivial fixed point $[\beta(g_*) = 0]$ in the physical region $g > 0$ with the coordinate

$$g_* = 8(d+2)\varepsilon/3(d-1)\bar{S}_d + O(\varepsilon^2). \tag{2.25}$$

The correction exponent $\omega = \beta'(g_*) = 2\varepsilon + O(\varepsilon^2) > 0$ at this point is positive, so that it is IR attractive and governs the IR behaviour of the correlation functions. The value of $\gamma_\nu(g)$ at the fixed point is found exactly using relations (2.20):

$$\gamma_\nu^* \equiv \gamma_\nu(g_*) = 2\varepsilon/3. \tag{2.26}$$

Relations (2.23) and (2.24) illustrate the fact that the coefficients that have a well-defined finite limit for $d \to \infty$ are $b_{n1} = a_{n1}\bar{S}_d^{-n}$ rather than a_{n1} themselves (this is obvious from the practical calculation, see section 3). It is convenient to introduce the new coupling constant by the relation

$$u = g\bar{S}_d. \tag{2.27}$$

Then the correlation functions G and G^R, renormalization constants (2.18) and the RG functions γ_ν and $\beta = \widetilde{\mathcal{D}}_\mu u$ will have well-defined finite limits $d \to \infty$ when expressed as perturbation series in the parameter u; the latter is kept finite in that limit.

In the following, the large-d limit will always be understood in this sense. From the physics viewpoints it corresponds to the natural requirement that the quantity which is kept finite for $d \to \infty$ is the energy input per one component of the velocity field (that is, per one spatial dimension), which is clear from the relations (2.7), (2.8) and (2.15). Such a choice is also consistent with further RG analysis which shows that the fixed-point value u_* of the parameter u (and not that of g) is finite for $d \to \infty$, as illustrated by equation (2.25).

3. Calculation of the renormalization constant and RG functions to the third order

Let us turn to the calculation of the renormalization constant Z_ν in (2.18) with the accuracy $O(g^3)$ (three-loop approximation) in the limit $d \to \infty$. In general, the renormalization constant Z_Γ corresponding to a certain 1-irreducible Green function $\Gamma = \langle \Phi \cdots \Phi \rangle_{\text{1-ir}}$ can be found from relation [34] (see also the monograph [3])

$$Z_\Gamma = 1 - \mathcal{K}\mathcal{R}'\tilde{\Gamma}, \tag{3.1}$$

where $\tilde{\Gamma}$ is the function Γ normalized with respect to unity in the zeroth order of the perturbation theory, $\mathcal{R}'$ is the incomplete $\mathcal{R}$-operation which involves all the subtractions for the divergent subgraphs, without the last subtraction for the diagram as a whole, and $\mathcal{K}$ is the subtraction operation for the given renormalization scheme. In the MS scheme, which we use in our calculation, $\mathcal{K}$ subtracts only poles in ε:

$$\mathcal{K} \sum_{n=-\infty}^{\infty} a_n \varepsilon^n = \sum_{n=-\infty}^{-1} a_n \varepsilon^n$$

for any Laurent series. The function $\tilde{\Gamma}$ in (3.1) should be calculated in terms of the renormalized parameters from the 'basic' action functional [3, 35] which is obtained from the renormalized action (2.17) by the replacements $Z_i \to 1$ for all the renormalization constants.

The only independent renormalization constant Z_ν in our model (2.17) is determined by the 1-irreducible function (in the frequency–momentum representation)

$$\Gamma_{ij}(\omega, \mathbf{p}) = \langle v_i' v_j \rangle_{\text{1-ir}} = \Gamma(\omega, \mathbf{p}) P_{ij}(\mathbf{p}), \tag{3.2}$$

where the projector $P_{ij}(\mathbf{p}) = \delta_{ij} - p_i p_j / p^2$ arises due to the transversality of the fields v_i', v_j. Thus the scalar coefficient $\Gamma(\omega, \mathbf{p})$ in (3.2) is given by the relation

$$\Gamma(\omega, \mathbf{p}) = \Gamma_{ij}(\omega, \mathbf{p}) P_{ij}(\mathbf{p})/(d-1) = \Gamma_{ii}(\omega, \mathbf{p})/(d-1), \tag{3.3}$$

where $(d-1) = P_{ii}(\mathbf{p})$ comes from the trace of the projector. In the perturbation theory,

$$\Gamma(\omega, \mathbf{p}) = \mathrm{i}\omega - \nu p^2 + O(g).$$

Since the counterterm to the function (3.2) is proportional to νp^2, one can put $\omega = 0$ in Γ and neglect the higher-order terms in p^2. Therefore the normalized scalar function $\tilde{\Gamma}$ in (3.1) can be taken as

$$\tilde{\Gamma} = \lim_{\mathbf{p} \to 0} \frac{\langle v_i' v_i \rangle_{\text{1-ir}}(\omega = 0; \mathbf{p})}{\nu p^2 (1-d)} = 1 + O(g), \tag{3.4}$$

it depends only on the completely dimensionless variables g and m/μ.

The elementary integrations over times (or equivalently frequencies) in the diagrams are always easily performed; the serious problem is only the integration over the momenta. (The factors ν arising from these integrations and from the propagator $\langle vv \rangle_0$ group together to cancel ν in the denominator of (3.4).)

The first important observation is that, for any diagram of the function (3.2), the number of loops (that is, the number of independent integration momenta) is equal to the number of lines (propagators) of the type $\langle vv \rangle_0$ (this is easy to understand from a few examples, given in this section, so that we shall not give the formal proof). It is therefore possible to assign the 'pure' independent momenta (which we shall denote by $\mathbf{k}$, $\mathbf{q}$ and $\mathbf{l}$ for the three-loop diagrams) to these propagators. Then the momenta associated with the remaining lines of the type $\langle vv' \rangle_0$ will be certain definite linear combinations of the external momentum argument $\mathbf{p}$ and the integration momenta $\mathbf{k}$, $\mathbf{q}$ etc.

Each line $\langle vv \rangle_0$ brings about one integration over an independent pure momentum:

$$\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \langle vv \rangle_0(\mathbf{k}) \cdots = \frac{g\mu^{2\varepsilon}\nu^2}{(2\pi)^d} \int \mathrm{d}\mathbf{k}\, k^{2-d-2\varepsilon} \cdots. \tag{3.5}$$

Here we used the last expression from (2.13) and omit the dimensionless time-dependent factor and the projector, denoted by the ellipsis. Now we can separate the integration over $\mathbf{k}$ into the integrations over the modulus $k = |\mathbf{k}|$ and the direction $\mathbf{n} = \mathbf{k}/k$ ('angles'), which gives $\mathrm{d}\mathbf{k} = k^{d-1}\mathrm{d}k\,\mathrm{d}\mathbf{n}$, and replace the integration over the angles by the averaging $\langle \cdots \rangle_S$ over the unit d-dimensional sphere, normalized such that $\langle 1 \rangle_S = 1$. This gives

$$\frac{g\mu^{2\varepsilon}\nu^2}{(2\pi)^d} \int_m^\infty \mathrm{d}k\, k^{1-2\varepsilon} \int \mathrm{d}\mathbf{n} \cdots = g\mu^{2\varepsilon}\nu^2 \bar{S}_d \int_m^\infty \mathrm{d}k\, k^{1-2\varepsilon} \langle \cdots \rangle_S. \tag{3.6}$$

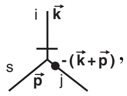
The factor $\bar{S}_d$ from (2.7) groups together with the coupling constant g; this quantity is kept fixed in the limit $d \to \infty$. The lower limit m in the integration over k comes from (2.10) and provides the IR regularization.

In (3.6) the powers k^{-d} from the correlator (2.9) and k^d from the Jacobian cancel each other, so that the dependence on d in the exponent disappears. The remaining dependence on d comes from the contractions of the projectors (2.13) and vertices (2.14), with further integrations over the angles.

The vertex V^S_{ijs} in (2.14) is explicitly symmetric with respect to the indices js, that is, symmetric with respect to the two attached fields v_j and v_s. Let us split it into the two asymmetric parts

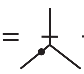
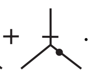
$$V^S_{ijs} = V^A_{ijs} + V^A_{isj}, \tag{3.7}$$

where the new asymmetric vertex $V^A_{ijs} = \mathrm{i}k_j\delta_{is}$ in the diagrams will be denoted as

$$V^A_{ijs} = \mathrm{i}k_j\delta_{is} = -\mathrm{i}p_j\delta_{is} =$$ 

, (3.8)

so that the splitting (3.7) is diagrammatically represented as

$V^S_{ijs} =$  $=$  $+$  . (3.9)

Here, the end marked with the slash corresponds to the field v', the end with no marks corresponds to the field v which stands under the derivative in the expression $-v'(v\partial)v = v'_i V^A_{ijs} v_j v_s$; the vector indices of these two fields are contracted to each other. The end marked with the thick dot corresponds to the second field v (without a derivative); its vector index is contracted with the index of the momentum. The momentum $\mathbf{k}$ is the argument of the field v'. In the asymmetric vertex, it can be equally replaced with $-\mathbf{p}$, the argument (up to the minus sign) of the unmarked field v, due to the transversality of the marked field v.

The use of the asymmetric vertex leads, of course, to increase in the number of diagrams: each diagram with N symmetric vertices (2.14) gives rise to the sum of 2^N contributions, represented by diagrams with vertices (3.8), as illustrated in figure 1 for the one-loop case. This inconvenience is compensated by the fact that the large-d behaviour of such individual contributions is easier to analyse. We shall see below that most of them vanish in the limit $d \to \infty$, and it is not possible to draw the corresponding diagrams from the very beginning.

From (3.3) it follows that a diagram can survive in the limit $d \to \infty$ only if its contribution to the quantity $\Gamma_{ii}(\omega, \mathbf{p})$ in the numerator behaves as $O(d)$ for large d. The only possible source of such a contribution is a closed (self-contracted) chain of intermittent δ-symbols,

$$\delta_{ii_1}\delta_{i_1i_2}\cdots\delta_{i_{n-1}i_n}\delta_{i_nj}\delta_{ji}, \tag{3.10}$$

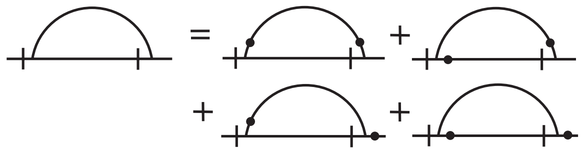


Figure 1. Passage to the asymmetric vertex in the one-loop approximation.

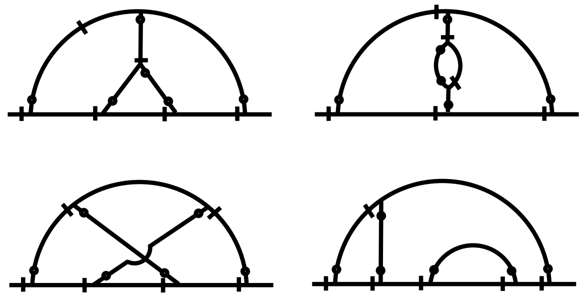


Figure 2. Examples of three-loop diagrams which do not vanish for $d \to \infty$.

coming from the vertices (3.8) and from the projectors in propagators (2.13), with the final contraction with δ_{ji} from the projector in (3.3). It is easily seen that, in terms of the asymmetric vertex, each diagram can involve no more than only one chain of the type (3.10). Assume for definiteness that the factor δ_{ii_1} in (3.10) comes from the leftmost vertex of the diagram, then $\delta_{i_n j}$ comes from the rightmost vertex. Then from the form of the vertex (3.8) it follows that the vector index i corresponds to the field v with a derivative, and i_1 corresponds to the field v'. The latter is necessarily contracted with the field v in the neighbouring vertex, because the propagator $\langle v'v'\rangle_0$ in (2.13) vanishes identically. Thus the next factor $\delta_{i_1 i_2}$ comes from the propagator $\langle v'v\rangle_0$. It is contracted with the next symbol $\delta_{i_2 i_3}$ from the vertex, in which i_2 must correspond to the field v with a derivative (any contraction with the momentum in (3.8) would break the chain of δ symbols); then i_3 necessarily corresponds to the field v', and so on.

We may conclude that the product of the type (3.10) is necessarily associated with the chain of propagators $\langle v'v\rangle_0$ which begins in the leftmost vertex of the diagram and ends up in the rightmost vertex; we shall call it 'the spine' in the following[1]. It can be formally proved, but is easily seen from the examples, that each diagram of the 1-irreducible function (3.2) involves one and only one spine. However, the spine gives rise to the factor of the type (3.10) only if all the fields v enter it with derivatives, so that their vector indices are contracted with the indices of the fields v'; this means that all the dots in the vertices (3.8) must be placed out of the spine. This fixes the placements of the dots in the vertices that belong to the spine in the unique way and drastically reduces the number of relevant diagrams. In figure 1, only the first diagram contains the factor (3.10) and gives a nonvanishing contribution to the quantity (3.3); the other diagrams vanish for $d \to \infty$ and can be dropped from the very beginning. Figure 2 gives examples of nonvanishing three-loop diagrams with the proper placement of the dots on the spine (here and in the figures below, the spine is always shown as a horizontal chain of propagators).

[1] The factor of the form (3.10) can also be associated with a *closed* circuit of the propagators $\langle v'v\rangle_0$, not connected to the external vertices of the diagram. But such diagrams vanish identically owing to the fact that the propagator $\langle v'v\rangle_0$ is retarded: it contains the step function in the time arguments; see (2.13).

The next important observation is that any nontrivial angular integration brings about smallness in $1/d$; the corresponding contributions in the diagrams can be dropped for $d \to \infty$. More precisely, the average over the unit sphere of the scalar product of two independent momenta $\mathbf{k}$ and $\mathbf{q}$ with the angle ϑ between them is given by the relations

$$\langle (\mathbf{kq})^{2n} \rangle_S = (kq)^{2n} \langle \cos^{2n} \vartheta \rangle_S = (kq)^{2n} \frac{(2n-1)!!}{d(d+2)\cdots(d+2n-2)} = O(d^{-n}) \qquad (3.11)$$

and $\langle (\mathbf{kq})^{2n+1} \rangle_S = 0$; here $n = 1, 2$ and so on. In the three-loop diagrams, averages of the form

$$\langle (\mathbf{kq})^{n_1} (\mathbf{kl})^{n_2} (\mathbf{lq})^{n_3} \rangle_S \qquad (3.12)$$

arise in the integrands. The corresponding expressions, analogous to (3.11), are rather complicated; they can be found e.g. in section VIC of [17]. It is only important for us here that they always vanish for $d \to \infty$ (with the obvious trivial exception $n_i = 0$ for all i.)

In all the relevant diagrams, the momentum argument of the two external vertices can be chosen equal to $\mathbf{p}$, the external momentum argument of the function (3.2). (This would be impossible for the diagrams in which the dot in the rightmost external vertex is placed on the external line, like in the last two diagrams in figure 1. The momentum arguments for such vertices necessarily involve internal integration momenta $\mathbf{k}$, $\mathbf{q}$ and so on. But we have already established that for $d \to \infty$ their contributions vanish.) Thus the overall factor of the form $p_i p_j$ is isolated from the quantity (3.3), and, since we are interested in the limit (3.2), we can set $\mathbf{p} = 0$ in the rest of the integrand.

If one of the external momenta, say p_i, is contracted with some integration momentum in the integrand, say k_i or $q_i + k_i$, a construction with scalar products of the form (3.11) or (3.12) arises, and in the limit $d \to \infty$ the diagram vanishes after the angular integrations. Nonvanishing contribution arises only if the external momenta are contracted with each other through a chain of δ symbols:

$$p_i \delta_{ii_1} \delta_{i_1 i_2} \cdots \delta_{i_{n-1} i_n} \delta_{i_n j} p_j. \qquad (3.13)$$

In the diagram, it corresponds to the chain of propagators that connects the two external vertices, with only two dots placed on the chain at the external vertices; they correspond to the external momentum. Each dot placed on the chain at an internal vertex corresponds to contraction with some integration momentum $\mathbf{k}$, $\mathbf{q}$ etc, and therefore breaks the chain (3.13). All the diagrams shown in figure 2 have such 'second spine,' having the form of a big arc connecting the external vertices and surrounding the diagram from above; the proper placement of the dots is also shown. From these examples it is clear that the second spine always has the form

$$\langle vv' \rangle_0 \langle vv' \rangle_0 \cdots \langle vv' \rangle_0 \langle vv \rangle_0 \langle v'v \rangle_0 \cdots \langle v'v \rangle_0, \qquad (3.14)$$

that is, it consists of two chains of mixed propagators $\langle vv' \rangle_0$ and $\langle v'v \rangle_0$ (not necessarily of equal 'lengths'), proceeding from the opposite directions and connecting with the only propagator $\langle vv \rangle_0$. In the simplest case of the only nonvanishing one-loop diagram in figure 1, the 'second spine' exists and consists of the single propagator $\langle vv \rangle_0$.

If the 'second spine' exists, it is unique because the chain of δ symbols in (3.13) 'cannot branch'. But the existence is not guaranteed (in contrast to the 'first spine' discussed above), as illustrated by the diagrams in figure 3. The first spines, shown as horizontal lines, consist of five and three propagators, respectively (the proper placement of the dots, which gives rise to the chains of the form (3.10), is always unique and is not shown). But the second spines do not exist: for the first diagram, there is no chain of propagators, connecting the external vertices.

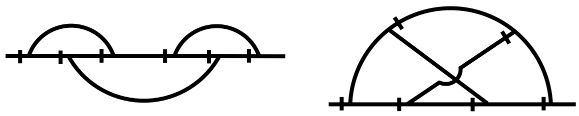


Figure 3. Examples of three-loop diagrams which have no 'second spine' and vanish for $d \to \infty$.

For the second diagram, such a chain exists (the big arc) but it involves *two* propagators $\langle vv \rangle_0$ and thus it does not have the form (3.14); any placement of the dot in the right upper vertex breaks the chain of δ symbols in (3.13).

Such diagrams give no contribution for $d \to \infty$, so they can be dropped from the very beginning. In the nonvanishing diagrams, the factor p^2 coming from the chain (3.13), explicitly cancels the analogous factor in the denominator of (3.4); the remaining quantity depends only on the dimensionless parameters g, ε and μ/m (the latter will also disappear soon).

Discarding the diagrams which vanish for $d \to \infty$ (those with 'wrong' placement of the dots on the spine or those having no second spine) leads to impressive decrease in the number of the surviving diagrams. In the one-loop approximation only one of the four diagrams survives (as is clear from figure 1), in the two-loop approximation 6 diagrams of 120 survive, and in the three-loop approximation only 80 diagrams survive of 8160.

The final simplification in the remaining nonvanishing diagrams is the possibility to drop all the scalar products of the integration momenta $\mathbf{k}$, $\mathbf{q}$ and so on, due to the smallness in the averages in expressions (3.11) and (3.12). If a scalar product appears in the denominator of the integrand, say $k^2 + q^2 + (\mathbf{kq})$, these expressions are not applied directly, but one can expand all the denominators in all the scalar products and then discard all the terms except for the first one (with no scalar products) due to relation (3.11) and its analogue for more complicated structures like (3.12). This is the same as to drop all the scalar products in the integrals from the very beginning. The diagrams in which the integrands are proportional to some scalar products vanish after this procedure, but we have no general rule how to identify them from the very beginning. The averaging over the angles now becomes trivial: $\langle 1 \rangle_S = 1$.

Thus the expressions for the remaining nonvanishing diagrams in the n-loop approximation are represented by n-fold integrals over the n scalar variables, the moduli of the independent integration momenta k, q and so on. It is convenient to pass to new integration variables k^2, q^2 and so on, then m^2 will appear as the new IR cutoff in the integrals. We shall retain the same notation k, q, m for these new parameters. Then, for $n = 3$, such integrals take on the forms

$$J(m) = \int_m^\infty \frac{\mathrm{d}k}{k^{1+\varepsilon}} \int_m^\infty \frac{\mathrm{d}q}{q^{1+\varepsilon}} \int_m^\infty \frac{\mathrm{d}l}{l^{1+\varepsilon}} f(k, q, l), \tag{3.15}$$

where f is some dimensionless function. From dimensionality considerations it is clear that the integral (3.15) is the definite power of the IR scale, $J(m) = m^{-3\varepsilon} J(1)$. Then applying the differential operation $\mathcal{D}_m = m\partial/\partial m$ to the integral (3.15) multiplies it by the factor (-3ε), and the following identity holds:

$$J(1) = -\frac{1}{3\varepsilon} \mathcal{D}_m J(m)|_{m=1}. \tag{3.16}$$

The parameter m enters expression (3.15) only in the lower integration limits, so that explicit differentiation on the right-hand side of (3.16) reduces the number of integrations by one. Since one pole in ε in (3.16) is already isolated explicitly, in the resulting double integrals one should retain the terms of order ε^{-2}, ε^{-1} and ε^0.

The complete three-loop calculation involves only six basic integrals J_i of the form (3.15) with the following functions f_i:

$$f_1 = \frac{kql}{(k+q+l)^3}, \qquad f_2 = \frac{kq}{(k+q+l)^2}, \qquad f_3 = \frac{kql}{(k+q)(k+l)(q+l)},$$
$$f_4 = \frac{kq}{(k+q)(k+q+l)}, \qquad f_5 = \frac{kq}{(k+q)(k+l)}, \qquad f_6 = \frac{kq}{(k+q)^2}. \tag{3.17}$$

The practical calculation gives

$$\tilde{J}_1 = \frac{1}{2}, \qquad \tilde{J}_2 = \frac{1}{2\varepsilon}, \qquad \tilde{J}_3 = \frac{\pi^2}{4}, \qquad \tilde{J}_4 = \frac{1}{2\varepsilon} + 1,$$
$$\tilde{J}_5 = \frac{1}{2\varepsilon^2} + \frac{\pi^2}{6}, \qquad \tilde{J}_6 = \frac{3}{2\varepsilon} - 3\ln 2, \tag{3.18}$$

where we denoted $J_i(1) = (1/3\varepsilon)\tilde{J}_i$.

The $\mathcal{R}$-operation in (3.1) implies subtraction of the UV divergences related to the subgraphs; this is a standard routine, see e.g. [3, 35]. It requires calculation of the one-loop and two-loop analogues of the integrals like (3.15) with the proper accuracy; this is of course an easier task than the calculation of the three-loop integrals. Combining all the contributions finally gives the following third-order result for the renormalization constant (2.18):

$$Z_\nu = 1 - \frac{u}{8\varepsilon} - \frac{u^2\,(2+\varepsilon)}{128\varepsilon^2} - \frac{u^3(10+9\varepsilon+7\varepsilon^2)}{3072\varepsilon^3} + O(u^4), \tag{3.19}$$

with the coupling constant u introduced in (2.27). It is worth noting that the transcendental numbers ($\ln 2$ and π^2) entering expressions (3.18) have disappeared in (3.19) as a result of the subtraction of subdivergences.

Then for the anomalous dimension (2.20) using relation (2.21) one finds

$$\gamma_\nu(u) = \tfrac{1}{4}u + \tfrac{1}{32}u^2 + \tfrac{7}{512}u^3 + O(u^4) \tag{3.20}$$

and the three-loop β function is found by the second relation in (2.20). Solving the equation $\beta(u_*) = 0$ perturbatively in ε, for the coordinate of the fixed point one obtains

$$u_* = \tfrac{8}{3}\varepsilon - \tfrac{8}{9}\varepsilon^2 - \tfrac{4}{9}\varepsilon^3 + O(\varepsilon^4). \tag{3.21}$$

The UV correction exponent

$$\omega = \beta'(u_*) = 2\varepsilon + \tfrac{2}{3}\varepsilon^2 + \tfrac{10}{9}\varepsilon^3 + O(\varepsilon^4) \tag{3.22}$$

completes the list of the third-order results.

The expressions obtained appear rather simple: in particular, all the coefficients in the series in g or ε for Z_ν, β, u_* and ω are rational numbers, although for dynamical models such coefficients usually involve transcendental quantities (logarithms, hypergeometric functions, etc) already in the two-loop approximations. One can hope that this fact remains valid for all the higher-order terms. It is then tempting to try to guess their general form and, therefore, to construct some hypothetical exact (that is, beyond the expansion in g or ε) expressions for those quantities. Such exact answers for all the quantities (3.19)–(3.22) exist [36] for the well-known Heisenberg model of the developed turbulence [37, 38]. In particular, for ω and u_* (in the MS scheme) one has [36]

$$\omega = \varepsilon(1-\varepsilon/3)/(1-5\varepsilon/12), \qquad u_* = (\varepsilon/3)(1-\varepsilon/3)^{1/4}, \tag{3.23}$$

where all the parameters have the same meaning as in our model (it is implied, of course, that the usual Heisenberg model is generalized to the energy pumping of the form (2.5)). Let us assume that in our model the fixed-point coordinate has a similar form:

$$u_* = (c_1\varepsilon)(1+\varepsilon c_2)^{c_3}.$$

Then comparison with the cut (3.21) of the ε expansion for u_* allows one to find all the coefficients c_i, which appear very simple and, surprisingly enough, the exponent $c_3 = 1/4$ is exactly the same as in the Heisenberg model:

$$u_* = (8\varepsilon/3)(1 - 4\varepsilon/3)^{1/4}. \tag{3.24}$$

Now the exact expression for ω can be found from (3.24). Differentiating equation (2.20) for the function $\beta = \widetilde{\mathcal{D}}_\mu u$ with respect to u, setting $u = u_*$ and taking into account $u_* \neq 0$ gives

$$\omega = \beta'(u_*) = 3\gamma_\nu'(u_*),$$

and differentiating the equation (2.26) with respect to ε gives

$$\gamma_\nu'(u_*)u_*'(\varepsilon) = 2/3$$

(the first differentiation with respect to u and the second—with respect to ε). Combining these equalities gives the exact relation

$$\omega(\varepsilon) = 2u_*/u_*'.$$

Substituting the explicit expression (3.24) for the function $u_* = u_*(\varepsilon)$ gives the desired hypothetical exact answer for the UV correction exponent:

$$\omega = 2\varepsilon(1 - 4\varepsilon/3)/(1 - 5\varepsilon/3), \tag{3.25}$$

which looks very similar to the corresponding expression (3.23) for Heisenberg's model and coincides (as is easily checked) with the expansion (3.22). From (3.24) and (3.25) it is then possible to derive the exact expressions for the β function, anomalous dimension γ_ν and the renormalization constant Z_ν; cf [36] for the Heisenberg model and [39] for the Kraichnan model. These expressions are rather cumbersome and will not be reported here.

Expressions (3.24), (3.25), provided they are correct, show that the IR fixed point $u_* > 0$ disappears at $\varepsilon = 3/4$, that is, before the physical value $\varepsilon = 2$ is achieved. This must lead to some qualitative change in the IR behaviour of the correlation functions which cannot be properly described within any finite-order approximation of the ε expansion. Of course, one should not take too seriously these hypothetical expressions, which, of course, are not the only ones possible.

4. Pair correlation function: third-order approximation

In this section we shall discuss the equal-time pair correlation function of the velocity field in the momentum representation,

$$D_{ij}(\mathbf{p}) = P_{ij}(\mathbf{p})D(p). \tag{4.1}$$

We are ultimately interested in its inertial-range behaviour ($m \ll p$), so in the following we set $m = 0$; the IR regularization in the diagrams is provided by the momentum p. Solving the RG equation (2.19) for the function $D(p)$ gives (see e.g. [3, 5, 6] for the detailed derivation)

$$D(p) = g\nu^2 p^{-d+2} R(s, g) \simeq D_0^{2/3} g_*^{1/3} p^{-d+2\Delta_v} R(1, g_*), \quad s = p/\mu. \tag{4.2}$$

The first equality, along with dimensionality considerations, introduces the function R which depends on the two dimensionless variables g and $s = p/\mu$ (dependence on ε and d is always implied). The second relation holds in the IR asymptotic region $s = p/\mu \ll 1$ and involves the coordinate of the IR attractive fixed point g_* and the critical dimension of the velocity field, which is known exactly: $\Delta_v = 1 - 2\varepsilon/3$. It depends only on the amplitude D_0 from (2.5) and is independent of the viscosity coefficient ν_0 in agreement with the second Kolmogorov hypothesis; see the discussion of this issue in [3, 5, 6, 11].

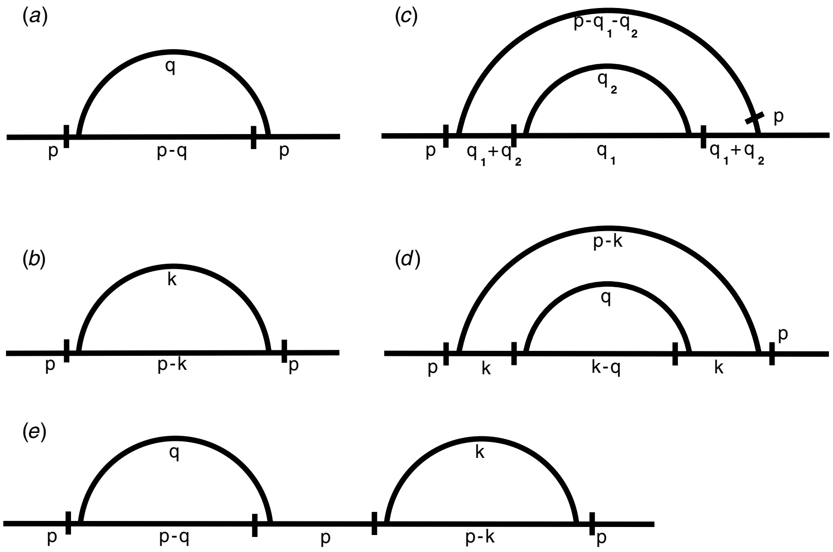


Figure 4. Examples of the one-loop and two-loop diagrams for the equal-time pair correlation function.

The function R can be directly calculated in the renormalized perturbation theory as a series in g, its coefficients being finite at $\varepsilon \to 0$. (One factor g in (4.2) is explicitly isolated from R such that its expansion in g begins with $O(g^0)$.) Substituting g_* as the series in ε (with the first-order term given by (2.25)) gives the ε expansion of the amplitude $R(1, g_*)$. It was calculated earlier in the one-loop approximation for certain values of d; see [10]. Here we shall calculate it in the limit $d \to \infty$ in the two-loop approximation (three terms of the ε expansion). This accuracy is consistent with the three-loop calculation of the renormalization constants and the coordinate $g_* \sim u_*$, presented in section 3.

The renormalized perturbation theory for the function (4.1) has the form

$$D(p) = \frac{g\nu^2}{2p^{d-2}} \left(\frac{\mu}{p}\right)^{2\varepsilon} \left\{1 + c_1 g \left(\frac{\mu}{p}\right)^{2\varepsilon} + c_2 g^2 \left(\frac{\mu}{p}\right)^{4\varepsilon} + \cdots \right\} \tag{4.3}$$

with some dimensionless coefficients $c_i = c_i(\varepsilon, d)$. The first-order term is the bare propagator $\langle v_i(t) v_j(t')\rangle_0$ from (2.13), in which we set $t = t'$ and replaced $\nu_0 \to \nu$ and so on. The contribution $D^{(n)}$ of a certain n-loop diagram to the series (4.3) has the form

$$D^{(n)}(p) = \frac{g^{n+1}\nu^2 A^{(n)}}{p^{d-2}} \left(\frac{\mu}{p}\right)^{2\varepsilon(n+1)}, \tag{4.4}$$

the problem is to calculate the dimensionless coefficient $A^{(n)} = A^{(n)}(\varepsilon, d)$. After the simple integrations over the time variables, the quantity (4.4) is represented as the integral over n independent momenta.

In contrast to the Green function (3.2), now it is impossible, in general, to assign these momenta to all the propagators $\langle vv\rangle_0$. The problem arises if the diagram includes a 1-irreducible subgraph of the type $\langle v'v'\rangle_{1\text{-ir}}$, which itself is not a subgraph of another 1-irreducible subgraph (then it depends as a whole on the external momentum $\mathbf{p}$). Then, as illustrated by figure 4, some of the propagators $\langle vv\rangle_0$ necessarily correspond to certain linear combinations of the independent momenta and the external momentum $\mathbf{p}$, and not to a 'pure' independent momentum. Thus the cancellation of the powers such as k^{-d} from the correlator (2.9) and k^d from the Jacobian (discussed for the Green function below the equation (3.6)) does not occur, the dependence on d in the exponent does not disappear, and the calculational techniques for $d \to \infty$, developed in the previous section, are not directly applicable.

The general situation is illustrated by specific examples given in figure 4. The external momentum is denoted by $\mathbf{p}$ and the independent integration momenta are $\mathbf{k}$, $\mathbf{q}$ and $\mathbf{q}_{1,2}$ (the difference between '$\mathbf{k}$-momenta' and '$\mathbf{q}$-momenta' will become clear a bit later). The problem of the non-cancellation does not occur for the diagrams a and c: the diagram a has no subgraphs of the type $\langle v'v'\rangle_{1\text{-ir}}$, while the diagram c has one such subgraph, but it belongs to another subgraph. Thus it is possible to assign the pure integration momenta (denoted by $\mathbf{q}$) or the external momentum $\mathbf{p}$ to all the $\langle vv\rangle_0$ lines, as shown in the figure. The diagrams b, d and e involve one 'harmful' subgraph of the form $\langle v'v'\rangle_{1\text{-ir}}$, and one of the $\langle vv\rangle_0$ lines necessarily corresponds to certain linear combination of 'pure' momenta; in all the examples, it is $\mathbf{p}-\mathbf{k}$. These examples also illustrate the general fact that any diagram can involve no more than one 'harmful' $\langle v'v'\rangle_{1\text{-ir}}$ subgraph. What is more, as illustrated by the diagram e, the problem of the non-cancellation occurs only within this subgraph; the other subgraphs involve only momenta of the type $\mathbf{q}$.

In principle, the problem can be attacked by the steepest descent method, but in practice it is hardly possible to find the stationarity manifold in the integration region. In order to circumvent this difficulty, we will apply the following trick.

Assume that the diagram indeed contains one 'harmful' subgraph of the type $\langle v'v'\rangle_{1\text{-ir}}$. Since the p-dependence of the quantity $D^{(n)}(p)$ in (4.4) is known, we can integrate the both sides of that equation over $\mathbf{p}$ with an arbitrary 'weight function' $F(p^2)$ (the only requirement is that the integral be convergent) and then extract the amplitude $A^{(n)}$ from the identity

$$g^{n+1}\mu^{2\varepsilon(n+1)}\nu^2 A^{(n)} = \frac{\int \mathrm{d}\mathbf{p}\, D^{(n)}(p)F(p^2)}{S_d\int_0^\infty \mathrm{d}p\, p^{1-2\varepsilon(n+1)}F(p^2)}. \tag{4.5}$$

In the denominator we have performed the angular integrations, which gives the factor S_d from (2.7). The numerator is represented by the integral over $(n+1)$ momenta: $\mathbf{p}$ and n independent momenta from the original diagram. As is clear from the examples in figure 4, the number of the lines $\langle vv\rangle_0$ in any n-loop diagram is equal to $(n+1)$. Thus it is possible to perform the change of variables in the integral such that the pure independent integration momenta will be assigned to all the $\langle vv\rangle_0$ propagators. Now the powers of the integration momenta k^{-d} from the correlators (2.9) and k^d from the Jacobians cancel each other, the remaining dependence on d comes only from the contractions of projectors and vertices with further angular integrations. Thus we arrive at the integrals of the type discussed in the previous section, and they can be calculated for $d \to \infty$ in the same manner. In particular, all the scalar products can be dropped, and the angular integrals in the numerator of (4.5) give only the factor S_d^{n+1}; we are left with an $(n+1)$-fold integral over the moduli.

Let us derive a more explicit expression for the resulting integrals and check their independence of the weight function $F(p^2)$. In the new variables the former external momentum is certain linear combination of the part of the new independent momenta:

$$\mathbf{p} = \mathbf{k}_1 + \mathbf{k}_2 + \cdots + \mathbf{k}_l \tag{4.6}$$

with some $l \geqslant 2$. The remaining $m = n - l + 1$ independent momenta will be denoted as $\mathbf{q}_1, \ldots, \mathbf{q}_m$. Analysis of the diagrams shows that m is equal to the number of the 1-irreducible subgraphs of the form $\langle v'vv\cdots v\rangle_{1\text{-ir}}$, that is, subgraphs with one field v' and arbitrary number of the fields v. This is easily checked for the examples given in figure 4: $n = l = m = 1$

for the diagram b and $n = l = 2, m = 1$ for the diagrams d and e.[2] It is also clear that the momenta $\mathbf{q}_i$ here, are exactly the same as in the original variables, while $\mathbf{k}_i$ in (4.6) are certain linear combinations of $\mathbf{p}$ and the original momenta of the type $\mathbf{k}$.

Let us introduce the l-dimensional 'vector' $\mathbf{K} = \{k_1, k_2, \ldots, k_l\}$ (where $k_i = |\mathbf{k}_i| \geqslant 0$ for all $i = 1, \ldots, l$), the corresponding unit vector $\mathbf{n} = \mathbf{K}/K$, and pass to the spherical coordinates:

$$\int_0^\infty \mathrm{d}k_1 \cdots \int_0^\infty \mathrm{d}k_l = \int_0^\infty \mathrm{d}K\, K^{l-1} \int \mathrm{d}\mathbf{n}. \tag{4.7}$$

The integrations over the directions of the vector $\mathbf{n}$ are restricted by the inequalities $n_i \geqslant 0$. For $d \to \infty$, the scalar products can be dropped, so that

$$p^2 = (\mathbf{k}_1 + \mathbf{k}_2 + \cdots + \mathbf{k}_l)^2 \simeq k_1^2 + k_2^2 + \cdots + k_l^2 = K^2.$$

The remaining variables can be made dimensionless by the replacement $q_i \to w_i = q_i/K$ ($i = 1, \ldots, m = n - l + 1$). In the new set of variables $\mathbf{n}$, w_i and K only the latter is dimensional. Thus the dependence of the integrand on K is found from the dimensional considerations as a known power, the integral over K factorizes out from the remaining integrals in the form

$$I_n = \int_0^\infty \mathrm{d}K\, K^{1-2\varepsilon(n+1)} F(K^2) \tag{4.8}$$

and explicitly cancels the integral in the denominator of (4.5). This confirms the independence of the result on the choice of the weight function $F(K^2)$.

Thus we are left with some integrals over the dimensionless variables $\mathbf{n}$ and w_i of the general form

$$\int_{n_i \geqslant 0} \mathrm{d}\mathbf{n} \int \mathrm{d}w_1 \cdots \int \mathrm{d}w_m\, \psi(\mathbf{n}, w_1, \ldots, w_m) \tag{4.9}$$

with a known dimensionless function ψ. They appear rather simple: all the two-loop integrals are calculated exactly (that is, without the expansion in the Laurent series in ε) and are expressed in terms of Γ functions, trigonometric functions and simpler quantities.

From the calculation procedure described above one can conclude that in the limit $d \to \infty$, the momenta of all the propagators $\langle vv \rangle_0$ (but *not* those of the mixed propagators $\langle vv' \rangle_0$!) behave as if they were orthogonal to each other, including the case when one of such propagators corresponds to the external momentum $\mathbf{p}$. For any given diagram it can be checked *a posteriori* that the condition that all such momenta be orthogonal defines the stationarity manifold in the integration region in the steepest-descent calculation for $d \to \infty$. However, it would be difficult to derive the orthogonality condition starting from the stationarity equations, especially in the original variables, where some of the propagators $\langle vv \rangle_0$ correspond to linear combination of 'pure' integration momenta. Thus the trick described above appears extremely useful.

Let us illustrate the general scheme by the explicit calculation of the diagram d from figure 4. For this diagram $n = l = 2$ and $m = 1$ and, in agreement with the general statements made above, it involves two momenta of the type $\mathbf{k}$ in (4.6) and one momentum $\mathbf{q}$. All these momenta are assigned to the lines $\langle vv \rangle_0$ as shown in figure 5; the remaining lines correspond to their linear combinations. The dotted line denotes the weight function $F(p^2)$, where the former external momentum is represented as $\mathbf{p} = \mathbf{k}_1 + \mathbf{k}_2$.

[2] This statement does not apply to the graphs like a and c in figure 4, which have no 'harmful' $\langle v'v' \rangle_{1\text{-ir}}$ subgraph and involves no momenta of the type $\mathbf{k}$; the trick is not needed for such diagrams.

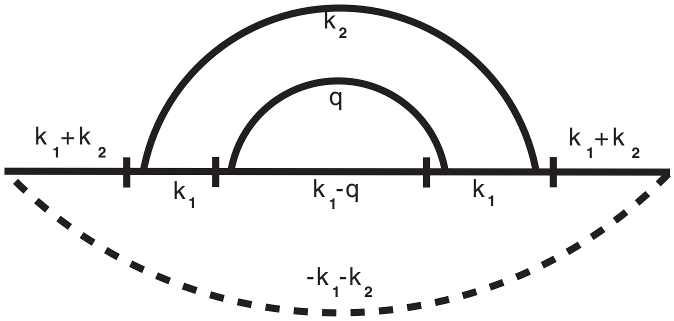


Figure 5. The new choice of the independent integration momenta in a two-loop diagram with the 'harmful' subgraph.

The integrand in the numerator of expression (4.5) has the form

$$\left(-k_1^2K^2\right)\times\frac{3}{16K^4}\left\{\frac{2}{k_1^2\left(k_1^2+q^2\right)}+\frac{1}{\left(k_1^2+q^2\right)(K^2+q^2)}+\frac{1}{K^2(K^2+q^2)}\right\}$$
$$\times\frac{1}{8(k_1k_2q)^{d-2+2\varepsilon}},\qquad K^2=k_1^2+k_2^2.$$

Here the first factor comes from the vertices (3.8), the second one—from the time integrations and the third one—from the propagators $\langle vv\rangle_0$ in (2.13). After the angular integrations the integral in the numerator of expression (4.5) takes on the form

$$J=-\frac{S_d^3}{128(2\pi)^{2d}}\int_0^\infty \mathrm{d}k_1\int_0^\infty \mathrm{d}k_2\int_0^\infty \mathrm{d}q\,\frac{k_1^2}{K^2(k_1k_2q)^{2\varepsilon-1}}$$
$$\times\left\{\frac{2}{k_1^2\left(k_1^2+q^2\right)}+\frac{1}{\left(k_1^2+q^2\right)(K^2+q^2)}+\frac{1}{K^2(K^2+q^2)}\right\}F(K^2). \tag{4.10}$$

Passing to the new variables $k_i=Kn_i$, $q=Kw$ gives

$$J=\frac{S_d^3}{128(2\pi)^{2d}}\tilde{J}I_0,$$

where the integral

$$I_0=\int_0^\infty \mathrm{d}K\,K^{1-2\varepsilon}F(K^2)$$

from (4.8) cancels out the denominator in (4.5) and the remaining integral $\tilde{J}$ has the form

$$\tilde{J}=-\int_{n_i\geqslant 0}\mathrm{d}\mathbf{n}\int_0^\infty \mathrm{d}w\,\frac{n_1^2}{(n_1n_2w)^{2\varepsilon-1}}\left\{\frac{2}{n_1^2\left(n_1^2+w^2\right)}+\frac{1}{\left(n_1^2+w^2\right)(1+w^2)}+\frac{1}{(1+w^2)}\right\}. \tag{4.11}$$

In the spherical coordinates $n_1=\cos\theta$ and $n_2=\sin\theta$ the integral over the directions in (4.11) takes on the form

$$\int_{n_i\geqslant 0}\mathrm{d}\mathbf{n}=\int_0^{\pi/2}\mathrm{d}\theta.$$

The remaining integrations over w and θ are performed explicitly and give the following final answer for the diagram in figure 5:

$$\tilde{J} = \Gamma(\varepsilon)\Gamma(-\varepsilon)\Gamma(1-\varepsilon)\left\{\frac{(4\varepsilon-1)\Gamma(1-2\varepsilon)}{2\Gamma(2-3\varepsilon)} + \frac{(2-\varepsilon)\Gamma(1-\varepsilon)}{4\Gamma(2-2\varepsilon)}\right\}. \tag{4.12}$$

The explicit result (4.12) illustrates some general properties of the diagrams of the pair correlator (4.1). In general, the diagrams contain poles at $\varepsilon = 0$ (manifestation of the UV divergences) which are cancelled by the poles coming from the renormalization constants in the action (2.17) in every order of the renormalized perturbation theory, so that the function R in (4.2) is finite at $\varepsilon \to 0$ in every order of the expansion in u. In addition, expression (4.12) involves the poles for some finite values of ε, namely $\varepsilon = 1/2, 1, 3/2$ and 2. All the other two-loop diagrams contain the poles at $\varepsilon = 1$ and 2 and some more diagrams contain the poles at $\varepsilon = 1/2$ and $3/2$. Such singularities are manifestation of the IR divergences (that is, divergences coming from the region of small integration momenta) that occur in the diagrams if the calculation is performed in the 'massless' ($m = 0$) model. When the IR cut-off $m > 0$ in (2.9) is restored, the diagrams become convergent for all $\varepsilon > 0$, and the IR problem is reformulated as the analysis of the singularities at $m \to 0$. The problem of such singularities in the stochastic NS equation with the stirring force (2.9) was discussed in a number of studies; see e.g. [7–12].

The one-loop diagrams only contain the first-order poles at $\varepsilon = 1$ and 2. As illustrated by the present two-loop calculation, in the higher-order diagrams the order of such poles increases and they are getting closer and closer to the origin $\varepsilon = 0$. Thus the problem of the IR singularities lies beyond the scope of any given-order approximation of the RG, let alone the ordinary perturbation theory. For the correct analysis of the small-m behaviour, the standard RG techniques should be combined with additional methods—infrared perturbation theory or short-distance operator-product expansion [10–12]. For small values of ε, this approach allows one to prove the existence of the limit $m \to 0$, which justifies the expansion in ε of the scaling amplitude $R(1, u_*)$ in (4.2).[3]

The final result for the scaling function in (4.2) has the form

$$R(1, u) = \frac{1}{2} + \frac{u}{16}\left\{\frac{1}{2} + \varepsilon\left(1 - \frac{\pi^2}{6}\right)\right\} + \frac{u^2\pi^2}{384} + \cdots, \tag{4.13}$$

up to the terms of higher orders in $u \sim \varepsilon$. Substituting the fixed-point coordinate (3.21) gives

$$R(1, u_*) = 1/2 + \varepsilon/12 + (5/36 - \pi^2/108)\varepsilon^2 + O(\varepsilon^3). \tag{4.14}$$

This is the main result of the present section; it will be used below in the calculation of the Kolmogorov constant and the skewness factor. Although only the first two terms of the coordinate (3.21) are involved in the derivation of (4.14), the $O(\varepsilon^3)$ term will be needed to calculate the three terms of the whole expression (4.2), which contains the product $u_*^{1/3} R(1, u_*)$. In this sense, the two-loop calculation of the pair correlator is consistent with the three-loop calculation of the RG functions.

5. Two-loop calculation of the Kolmogorov constant and skewness factor

The Kolmogorov constant C_K can be defined as the dimensionless coefficient in the inertial-range asymptotic expression $S_2(r) = C_K(\mathcal{E}r)^{2/3}$ for the second-order structure function,

[3] More physically, the IR singularities at $\varepsilon = 1/2$ and 1 are related to the so-called sweeping effects and must disappear in the sum of all the diagrams for the equal-time correlator, which is a Galilean-invariant quantity; they do. The singularities at $\varepsilon = 3/2$ and 2 do not disappear; they can be related to the fluctuations of the energy flux. These interesting issues will be discussed elsewhere.

predicted by the classical phenomenological Kolmogorov–Obukhov theory [1, 38].[4] Here $\mathcal{E}$ is the average energy dissipation rate (cf equation (2.4)) and the nth order (longitudinal, equal-time) structure function is defined as

$$S_n(r) = \langle [v_r(t, \mathbf{x}+\mathbf{r}) - v_r(t, \mathbf{x})]^n \rangle, \qquad v_r = (v_i r_i)/r, \quad r = |\mathbf{r}|. \tag{5.1}$$

Using this definition, the function S_2 can be related to the momentum–space pair correlation function $D(k)$ from (4.1) as follows:

$$S_2(r) = 2\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} D(k)[1 - (\mathbf{kr})^2/(kr)^2]\{1 - \exp[\mathrm{i}(\mathbf{kr})]\}. \tag{5.2}$$

Alternatively, the Kolmogorov constant C'_K can be introduced through the phenomenological relation $E(k) = C'_K \mathcal{E}^{2/3} k^{-5/3}$, where the energy spectrum $E(k)$ is related to the function (4.1) as $E(k) = \bar{S}_d(d-1)k^{d-1}D(k)/2$. From the definitions one can derive the following relation between these two constants:

$$C_K = \frac{3\times 2^{1/3}\Gamma(2/3)\Gamma(d/2)}{(d+2/3)\Gamma(d/2+1/3)} C'_K, \tag{5.3}$$

cf [38] for $d = 3$. Using the exact relation $S_3(r) = -12\mathcal{E}r/d(d+2)$ that follows from the energy balance equation (see e.g. [1, 38] for $d = 3$), the constant C_K can be related to the inertial-range skewness factor $\mathcal{S}$ as follows:

$$\mathcal{S} \equiv S_3/S_2^{3/2} = -[12/d(d+2)]C_K^{-3/2}. \tag{5.4}$$

All these relations refer to real physical quantities in the inertial range, which in the stochastic model (2.1), (2.2) corresponds to $\varepsilon = 2$ and $m = 0$ in the random force correlator (2.9).

Much work has been devoted to derivation of the Kolmogorov constant within the RG approach; see e.g. [10, 11, 32, 41–43] and references therein. In order to obtain C_K, one usually combines the RG expression (4.2) for $D(k)$ or the analogous expression for S_2 by some relation between the physical parameter $\mathcal{E}$ and the amplitude D_0 in the random force correlator (2.9). In particular, in [41] the first-order term of the ε expansion for the pair correlator was combined with the so-called eddy-damped quasinormal Markovian approximation for the energy transfer function, taken directly at the physical value $\varepsilon = 2$. More elementary derivation, based on the exact relation (2.4) between $\mathcal{E}$ and the function $d_f(k)$ from (2.9) was given in [11]; see also [5, 6]. In spite of a reasonable agreement with the experiment, such derivations are not satisfactory from the theoretical viewpoints. Their common drawback is that any relation between $\mathcal{E}$ and D_0 is unambiguous only in the limit $\varepsilon \to 2$ (see equation (2.8)), so that the coefficients of the ε expansions for C_K can be made arbitrary; see e.g. the discussion in [32] and section 2.10 of [6].

This ambiguity is a consequence of the fact that the notion itself of the Kolmogorov constant has no unique extension to the nonphysical range $0 < \varepsilon < 2$. Furthermore, the experience from the RG theory of critical behaviour suggests that well-defined ε expansions can be written for universal quantities, such as critical exponents or normalized scaling functions, which do not involve bare parameters. The constant C_K extended to the range $0 < \varepsilon < 2$ as in [11, 41, 42] involves a bare parameter, D_0, and hence is not universal in this sense.

To circumvent these difficulties, an alternative derivation was proposed in [10] that does not involve any relation between D_0 and $\mathcal{E}$, relates C_K to a universal (in the sense of the

[4] Now it is generally believed that the real exponent deviates slightly from 2/3 due to the phenomenon of intermittency; see the discussion in [1] and references therein. However, the situation is not absolutely clear: some researchers argued that the existing experimental data are consistent with the '2/3 law' and the observed disagreement is due to the *corrections* (and not *deviations*) vanishing in the limit of infinite Reynolds number; see e.g. [40]. Now we are interested in the large-d limit, in which intermittency is expected to get weaker or to disappear; so we will accept the '2/3 law', which is also internally consistent within the calculation based on the ε expansion.

theory of critical behaviour) quantity, and leads (for $d = 3$) to reasonable agreement with experimental data. Below we adopt this derivation to our case, $d \to \infty$.

Consider the ratio

$$Q(\varepsilon) = \mathcal{D}_r S_2(r)/|S_3(r)|^{2/3} = \mathcal{D}_r S_2(r)/(-S_3(r))^{2/3} \tag{5.5}$$

with $\mathcal{D}_r = r\partial/\partial r$. We shall see below that it is universal and can be calculated in the form of a well-defined ε expansion. On the other hand, its value at $\varepsilon = 2$ determines the Kolmogorov constant and the skewness factor through the exact relations

$$C_K = [3Q(2)/2]\,[12/d(d+2)]^{2/3}, \qquad \mathcal{S} = -\,[3Q(2)/2]^{-3/2} \tag{5.6}$$

which follow from the definitions, relation (5.4) and the identity $\mathcal{D}_r r^\delta = \delta r^\delta$ for any δ.

From the RG equations one derives the analogues of the representation (4.2) for the structure functions entering the ratio (5.5):

$$S_3(r) = D_0 r^{-3\Delta_v} f_3(\varepsilon), \qquad \mathcal{D}_r S_2(r) = D_0^{2/3} r^{-2\Delta_v} f_2(\varepsilon). \tag{5.7}$$

Here $m = 0$, $\mu r \gg 1$, $0 < \varepsilon \leqslant 2$ and $\Delta_v = 1 - 2\varepsilon/3$, cf (4.2). The operation $\mathcal{D}_r$ introduced in (5.5) 'kills' the constant contribution $\langle v^2\rangle$ in S_2 which does not exist without an UV cutoff for $\varepsilon < 3/2$ (see below); in S_3 such contribution is absent.

It follows from (5.7) that the amplitude D_0 disappears from the ratio $Q(\varepsilon) = f_2/(-f_3)^{2/3}$, and the latter can be calculated in the form $Q(\varepsilon) = \varepsilon^{1/3} p(\varepsilon)$, where $p(\varepsilon)$ is a power series in ε. Our results for the fixed point from section 3 and for the pair correlator from section 4 allow us to find the first three terms of $p(\varepsilon)$. In this sense, one can speak about the third-order approximation for the Kolmogorov constant (previous attempts have mostly been confined with the first order, with the second-order exception of [10]). To avoid possible misunderstanding, we stress again that we did not intend to extend the definition of the physical quantities C_K and $\mathcal{S}$ to the whole interval $0 < \varepsilon < 2$ and to construct their ε expansions from the known expansion for $Q(\varepsilon)$. Instead, the latter is used to give the value of $Q(2)$, which, in its turn, determines C_K and $\mathcal{S}$ through relations (5.6) that make sense only for the real value $\varepsilon = 2$.

Applying the operation $\mathcal{D}_r$ to expression (5.2) gives

$$\mathcal{D}_r S_2(r) = 2\int \frac{d\mathbf{k}}{(2\pi)^d} D(k)[1 - (\mathbf{kr})^2/(kr)^2](\mathbf{kr})\sin(\mathbf{kr}). \tag{5.8}$$

In order to obtain the inertial-range form of the function $\mathcal{D}_r S_2(r)$ it is sufficient to substitute the asymptotic expression (4.2) into (5.8). A straightforward calculation gives

$$\mathcal{D}_r S_2(r) = \frac{2(d-1)\Gamma(2-2\varepsilon/3)}{(4\pi)^{d/2}\Gamma(d/2+2\varepsilon/3)} g_*^{1/3} R(1,u_*) D_0^{2/3}(r/2)^{-2\Delta_v} \tag{5.9}$$

with the amplitude $R(1, u_*)$ from (4.2). It is important here that the integral in (5.8) with the function (4.2) exists for all $0 < \varepsilon < 2$. It is the operation $\mathcal{D}_r$ that ensures its convergence: the original integral (5.2) would be UV divergent for $0 < \varepsilon < 3/2$.

The needed terms of the ε expansion for f_3 can be obtained from the direct perturbative calculation, similar to the calculation of the pair correlator in section 4, but it is more convenient to use the exact expression

$$S_3(r) = -\frac{3(d-1)\Gamma(2-\varepsilon)}{(4\pi)^{d/2}\Gamma(d/2+\varepsilon)} D_0(r/2)^{-3\Delta_\varphi} \tag{5.10}$$

which follows from the energy balance equation and generalizes the well-known Kolmogorov's '4/3 law' [1, 38] to general d and ε. Thus from equations (5.9) and (5.10) for the ratio $Q(\varepsilon)$ one obtains

$$Q(\varepsilon) = [4(d-1)u_*/9]^{1/3} A(\varepsilon) R(1,u_*), \tag{5.11}$$

where we excluded g_* in favour of $u_* = g_* \bar{S}_d$; the coefficient $A(\varepsilon)$ is given by

$$A(\varepsilon) = \frac{\Gamma(2-2\varepsilon/3)\Gamma^{1/3}(d/2)\Gamma^{2/3}(d/2+\varepsilon)}{\Gamma(d/2+2\varepsilon/3)\Gamma^{2/3}(2-\varepsilon)}. \tag{5.12}$$

Using the Stirling formula for the Γ functions one can show that

$$\frac{\Gamma^{1/3}(d/2)\Gamma^{2/3}(d/2+\varepsilon)}{\Gamma(d/2+2\varepsilon/3)} = 1 + O(1/d),$$

so that in the limit $d \to \infty$ one obtains

$$A(\varepsilon) = \frac{\Gamma(2-2\varepsilon/3)}{\Gamma^{2/3}(2-\varepsilon)} = 1 + \frac{1}{9}\{1-\pi^2/6\}\varepsilon^2 + O(\varepsilon^3). \tag{5.13}$$

Substituting expressions (3.21), (4.14) and (5.13) into (5.11) gives

$$Q(\varepsilon) = \frac{(4\varepsilon d)^{1/3}}{3}\left\{1 + \frac{\varepsilon}{18} + \left(\frac{49}{162} - \frac{\pi^2}{27}\right)\varepsilon^2 + O(\varepsilon^3)\right\}. \tag{5.14}$$

Now the Kolmogorov constant C_K and the skewness factor $\mathcal{S}$ are obtained from expressions (5.6), with the replacement $d(d+2) \to d^2$ in the first one. Thus in the leading order of the large-d asymptotic behaviour one obtains $C_K \propto 1/d$ and $\mathcal{S} \propto 1/d^{1/2}$. From relation (5.3) it then follows $C'_K \propto d^{1/3}$, in agreement with the earlier results derived within the direct interaction approximation [24] and within the RG approach [5, 6].

Although these results refer to the large-d limit, one can try to use them as some approximation to the real three-dimensional case. Substituting the third-order result (5.14) into the second relation from (5.6) and setting $\varepsilon = 2$ and $d = 3$ gives $\mathcal{S} \approx -0.73$, while the experimental value recommended in [38] is $\mathcal{S} \approx -0.28$. Of course, one should not have expected a better agreement. Surprisingly enough, the situation appears much better for the Kolmogorov constant. Substituting the first, second and third approximations from (5.14) into the first relation from (5.6), one obtains for $\varepsilon = 2$ and $d = 3$ the following results:

$$C_K^{(1)} \approx 1.75, \qquad C_K^{(2)} \approx 1.94, \qquad C_K^{(3)} \approx 1.50,$$

all of them in a reasonable agreement with the experimental estimate $C_K \approx 1.9$ recommended in [38]. One can speculate that the skewness factor which is related to the odd-order function S_3 and is a measure of anisotropy in the distribution of the velocity field is more sensitive to the spatial dimension than the Kolmogorov constant, related to the even-order function S_2.

6. Conclusion

We have accomplished the complete three-loop calculation of the main ingredients of the RG analysis of the d-dimensional stochastic Navier–Stokes equation (2.1), (2.2) in the limit $d \to \infty$: the renormalization constant Z_ν, the β function, the coordinate of the fixed point u_* and the ultraviolet (UV) correction exponent ω; these results are summarized in equations (3.19)–(3.22). We have also calculated in the large-d limit the correlation function of the velocity in the third order of the ε expansion (two-loop approximation).

These results allowed us to derive third-order answers for two interesting physical quantities: the Kolmogorov constant C_K in the spectrum of turbulent energy and the inertial-range skewness factor $\mathcal{S}$. The former is in a reasonable agreement with the existing experimental data for the real three-dimensional turbulence.

The successful analytic calculation in the third order appeared feasible due to drastic simplifications that we found in the large-d limit: in particular, most of the Feynman diagrams for the Green function in that limit vanish, and the remaining ones are reduced to relatively simple (and analytically calculable) integrals. Some more efforts and tricks were required to derive the third-order results for the pair correlation function of the velocity.

Although we did not succeed in finding the exact solution for $d = \infty$, the simplifications that occur in the calculations in that limit and the simple form of the obtained results suggest that this is not an impossible task. Based on the three-loop answers, we proposed some hypothetical expressions beyond the ε expansion which show what, in principle, such exact results may look like.

We believe that the present results, as well as the calculational techniques developed in their derivation, will be useful in the further attempts of the construction of the systematical $1/d$ expansion for the fully developed turbulence.

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